BSD over function fields

Arithmetic of function fields:

Number field	function fields
$\mathbb{Z}$	$\mathbb{F}_q[T]$
$\mathbb{Q}$	$\mathbb{F}_q(T)$
$K/\mathbb{Q}$ finite	$K/\mathbb{F}_q(T)$ finite / Curve $C/\mathbb{F}_q$
Place of K	Closed points of C
Fractional ideals	Divisors on C
Principal ideals	Principal on C

BSD over number fields:

$$L(E/K,s) = \prod_{p \text{ good}} \left(1 - a_p N(p)^{-s} + N(p)^{1-2s}\right)^{-1} \times \prod_{p \text{ bad}} \left(1 - a_p N(p)^{-s}\right)^{-1}$$

Conjecture: (BSD)

- $\mathrm{ord}_{s=1} L(E/K, s) = \mathrm{rk}(E/K)$
- $Ш_{E/K}$ is finite

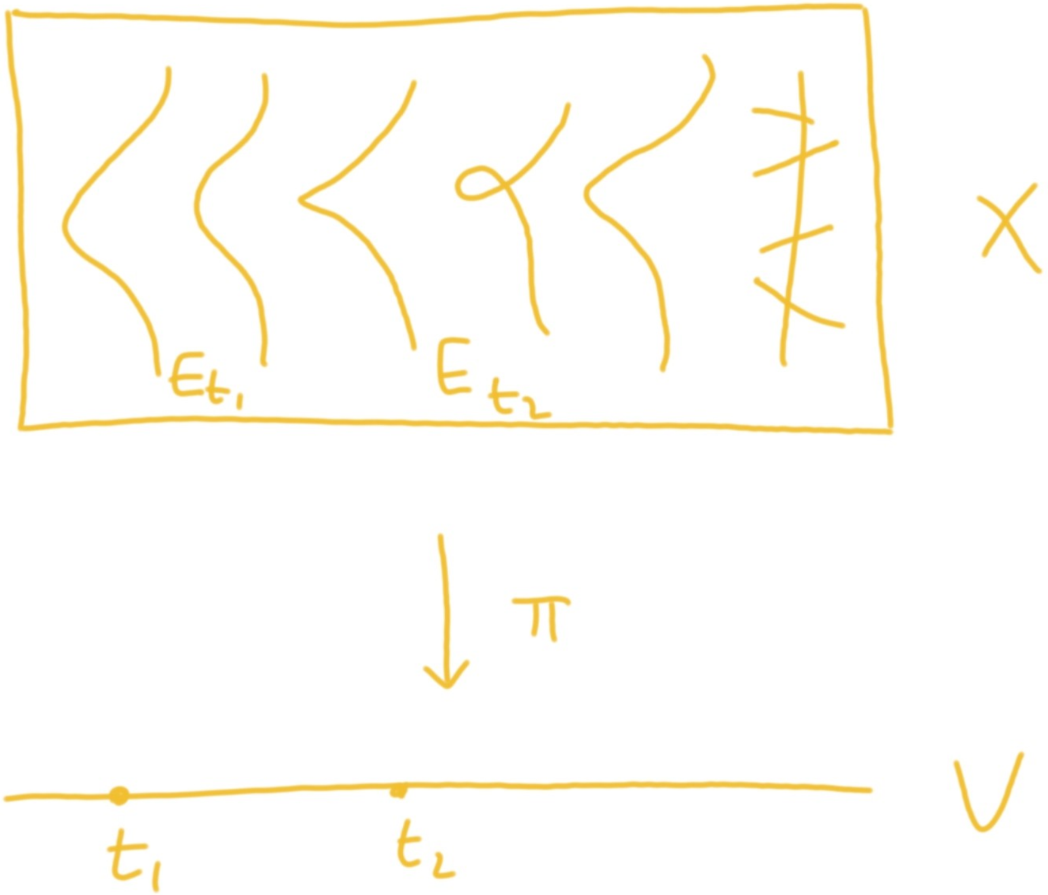
$$\lim_{s \to 1} \frac{L^*(E/K, s)}{(s-1)^r} = \frac{|Ш_{E/K}| \cdot \mathrm{Reg}_{E/K}}{|E(K)_{\mathrm{tors}}|^2}$$

Just replace everything using the analogy to get formulation for function fields.

Elliptic Surfaces:

$$E/\mathbb{F}_q(T): \quad y^2 = x^3 + a(T)x + b(T)$$

We can view this as a surface over $\mathbb{F}_q$, from which we should be able to recover.



In general we'll consider a surface $X/\mathbb{F}_q$ equipped with

$$X \xrightarrow{\pi} V$$

for some curve $V/\mathbb{F}_q$, w/ generic fibre $E/\mathbb{F}_q(V)$.

Q: Can we rephrase BSD as a statement about X ?

Weil Conjectures:

For ~~scheme of finite type~~ curve / surface $X/\mathbb{F}_q$ and consider

$$Z(X;T) = \prod_{P \text{ closed}} (1 - T^{\deg P})^{-1} = \exp\Big(\sum_{n \geq 1} \# X(\mathbb{F}_{q^n}) \frac{T^n}{n}\Big)$$

Theorem: $Z(X;T)$ is a rational function

$$Z(X;T) = \frac{P_1(T) \dots P_{2d-1}(T)}{P_0(T) \dots P_{2d}(T)}$$

such that $P_i(X;T) \in \mathbb{Z}[T]$ has complex roots w/ absolute value $q^{-i/2}$.

Example:
For an elliptic curve $\mathcal{E}/\mathbb{F}_q$, we have

$$Z(\mathcal{E};T) = \frac{1 - aT + qT^2}{(1-T)(1-qT)}.$$

Zeta and L-functions:

For an elliptic surface $X \xrightarrow{\pi} V$ we have

$$Z(X;T) = \prod_{P \text{ closed}} (1 - T^{\deg P})^{-1}$$

$$= \prod_{\substack{Q \in V \\ \text{closed}}} \prod_{P \in \pi^{-1}(V)} (1 - T^{\deg P})^{-1}$$

$$= \prod_{\substack{Q \in V \\ \text{closed}}} Z(\pi^{-1}(Q); T^{\deg Q}).$$

For smooth fibres, zeta-function looks as above so:

Theorem:

$$L(E|K,T) = \frac{Z(V;T)\cdot Z(V;qT)}{Z(X;T)} \times \prod_{Q \text{ bad}} \frac{(1-T)^{a_Q+1}(1+T)^{b_Q}}{(1-q_Q T^{\deg Q})^{f_Q-1}(1+q_Q T^{\deg Q})^{g_Q}}$$

where $f_Q = \#$ of components of $\pi^{-1}(Q)$ and rest also depends only on Kodaira type.

$$\operatorname{ord}_{T=1/q} L(E|K,T) = -\operatorname{ord}_{T=1/q} Z(X;T) - 2 - \sum_Q (f_Q - 1)$$

Néron - Severi:

$$NS(X) = \{\text{divisors}\}/\sim_{\text{algebraic}}$$

D_1 and D_2 are algebraically equivalent if they lie in a family parameterised by a smooth curve, i.e. if $\exists$ smooth curve $T/\bar{k}$, divisor D on T and $t_1, t_2 \in T$ such that $D_i = (X \times_{\bar{k}} \{t_i\}) \cap D$.

Linear equivalence $\Leftarrow$ Algebraic equivalence, so $NS(X)$ is a quotient of $Pic(X)$.

when $T = \mathbb{P}^1$

Facts: → $NS(X)$ is finitely generated.

→ $NS(X)$ inherits bilinear pairing from the intersection pairing on divisors.

Shioda–Tate:

Theorem: The Mordell-Weil rank of E is

$$\mathrm{rk}(NS(X)) - 2 - \sum_Q (f_Q - 1).$$

Define $\mathrm{Div}(X) \longrightarrow \mathrm{Div}(E)$ by sending C to its generic divisor $C \times_V E$. This is empty if C is supported in a fibre of π, or else is a closed point of E.

Extend linearly for a homomorphism, and write $\mathcal{L}^1 \mathrm{Div}(X)$ for the preimage of degree 0 divisors, and $\mathcal{L}^2 \mathrm{Div}(X)$ the kernel. Write $\mathcal{L}^1 NS(X)$ for the image in $NS(X)$.

Theorem: $$\mathcal{L}^1(NS(X)) / \mathcal{L}^2(NS(X)) = E(\mathbb{F}_q(V)).$$

Won't give a full proof, but notice that $E(\mathbb{F}_q(V))$ is at least contained in the image because we can choose as preimage the section S_P for $P \in E(\mathbb{F}_q(V))$.

Now $\mathrm{rk}(L^2 NS(X)) = 1 + \sum_Q (f_Q - 1)$:

Notice that fibres are algebraically equivalent via $T = V$, $D = \{(E_t, t) \in X \times V\}$. Then get the rest from singular fibres, but multiples of $\pi^{-1}(Q)$ already counted; sum over $f_Q - 1$.

$$\mathrm{rk}\, L^1(NS(X)) = \mathrm{rk}\, NS(X) - 1 = 1 + \sum (f_Q - 1) + \mathrm{rk}(E(\mathbb{F}_q(V))).$$

Geometric Analogue:

$$Ш_{E/K} = \ker\left(H^1(K, E) \longrightarrow \prod_v H^1(K_v, E_v)\right)$$

Non-trivial elements are isomorphism classes of twists with everywhere local but no global points.

Violations of Hasse principle $\rightsquigarrow$ $\mathrm{Br}(X)$.

Theorem: (Artin/Grothendieck?)

$$\mathrm{Br}(X) \simeq Ш_{E/\mathbb{F}_q(V)}$$

Now let ρ be the rank of $NS(X)$, and $\{D_i\}$ a basis for the free part:

Conjecture: (Artin – Tate)

(i) Multiplicity of $(1-qT)$ in $P_2(X;T)$ is ρ

(ii) $Br(X)$ is finite and

$$\frac{P_2(X;T)}{(1-qT)^{rk NS(X)}} = \frac{|Br(X)| \cdot |Det(D_i \cdot D_j)|}{q^{\alpha} \cdot |NS(X)_{tors}|^2},$$

where $\alpha \geqslant 0$ is an explicit fudge factor.

Theorem: (Artin – Tate, Milne)

The above conjecture is equivalent to BSD for E.

We have already seen this for ranks by combining Shioda – Tate with the relation between $Z(X;T)$ and $L(E/\mathbb{F}_q(U), T)$.

Rest of talk: showing the following theorem

Theorem: (Artin – Tate)

Suppose $\exists\, \ell$ s.t. $Br(X)(\ell)$ is finite. Then Artin – Tate conjecture holds up to sign and a power of p.

$$0 \to \mu_{\ell^n} \to \mathbb{G}_m \xrightarrow{\ell^n} \mathbb{G}_m \to 0$$

$\Big\downarrow$ cohomology

$$0 \to NS(\bar{X})/\ell^n \to H^2(\bar{X}, \mu_{\ell^n}) \to Br(\bar{X})_{\ell^n} \to 0$$

$\Big\downarrow$ inverse limit + Galois fixed space

We have an exact sequence

$$0 \longrightarrow NS(X) \otimes \mathbb{Z}_\ell \xrightarrow{h} H^2(X, T_\ell(\mu))^G \longrightarrow T_\ell(Br(X)) \longrightarrow 0$$

$\mu =$ sheaf of roots of unity

$G = Gal(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

so the following are equivalent:

(i) $Br(X)(\ell)$ is finite (ii) h is a bijection

(iii) $\rho = rk\ H^2(X, T_\ell(\mu))^G$.

Now we show (i), (ii), (iii) $\iff$ BSD (i) and also deduce BSD (ii) up to a factor.

Note that BSD (i) $\Rightarrow$ (iii) because

$$\det(\underline{1} - \Phi T \mid H^2(X, T_\ell(\mu)) \otimes \mathbb{Q}_\ell) = P_2(X; q^{-1}T)$$

so multiplicity of $1-qT$ in P_2 is the number of times 1 appears as an eigenvalue of Φ. This is at least $rk_{\mathbb{Z}_\ell} H^2(X, T_\ell \mu)^G$. (Because $\Phi \in G$ acts trivially on $(H^2)^G$).

Recall notation $f: A \longrightarrow B$ quasi-isom of f.g. $\mathbb{Z}_\ell$-modules

$$z(f) = \frac{|\mathrm{coker}(f)|_\ell}{|\ker(f)|_\ell}.$$

Facts: (i) If $\{a_i\}, \{b_j\}$ bases and $f(a_i) = \sum z_{ij} b_j$ then

$$z(f) = |\det(z_{ij})|_\ell \cdot \frac{|B_{tors}|_\ell}{|A_{tors}|_\ell}.$$

(ii) If $g: B \to C$, then if two of $f, g, g\circ f$ are quasi-isomophisms so is the third and

$$z(g\circ f) = z(g)\, z(f).$$

(iii) f quasi-isom. $\iff$ $f^\vee$ is and $z(f)z(f^\vee) = 1$.

(iv) Let ϑ be an endomorphism of A. Then the map $\ker\vartheta \xrightarrow{f} \mathrm{coker}\,\vartheta$ induced by id_A is a quasi-isom iff

$$\det(T - \vartheta \otimes_{\mathbb{Q}_\ell} 1) = T^{\mathrm{rk}\ker\vartheta} R(T)$$

w/ $R(0) \neq 0$. In this case $z(f) = |R(0)|_\ell$.

Think: $\vartheta = \Phi$ frobenius, $A = H^2_\ell(X)$

$\Rightarrow$: Consider commutative diagram

$$\begin{tikzcd} NS(X)\otimes \mathbb{Z}_\ell \arrow[r, "e"', "\text{intersection pairing}"] \arrow[d, "h"'] & \mathrm{Hom}(NS(X),\mathbb{Z}_\ell) \arrow[d, "g^*"] \\ H^2(X,T_\ell(\mu))^G \arrow[r, "f"] & H^2(X,T_\ell(\mu))_G \end{tikzcd}$$

where

$$g: NS(X)\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \longrightarrow (NS(\bar{X})\otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell)^G \hookrightarrow H^2(\bar{X},\mu_\ell)^G.$$

Fact (i) $\Rightarrow$ $\quad z(e) = \dfrac{|\det(D_i\cdot D_j)|_\ell}{|NS(X)[\ell]|_\ell}$.

(ii), (iii) $\Rightarrow$ $\quad z(g^*) = \dfrac{|NS(X)[\ell]|_\ell}{|Br(X)(\ell)|_\ell}$

Where $R(T) = \dfrac{P_2(X;T)}{(1-qT)^\rho}$, we have f is a quasi-isom so BSD (i) holds and:

$$|R(q^{-1})|_\ell = z(f) = z(e)z(g^*)^{-1} = \left|\frac{|Br(X)(\ell)|\cdot|\det(D_i\cdot D_j)|}{|NS(X)[\ell]|}\right|_\ell.$$

We have independence of ℓ so the theorem holds.

Final remarks:

All of the preceding works equally well for Jacobians of curves.

Theorem: (Kato–Trihan, 2003)

Let $A/\mathbb{F}_q(V)$ be an abelian variety over a global function field. If $\exists \ell$ s.t. $Ш_{A/\mathbb{F}_q(V)}(\ell)$ is finite, then BSD holds for A. That is,

BSD over function fields $\iff$ TS_ℓ for some ℓ.