Wild conductor exponents of curves

Harry Spencer

University College, London

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Wild conductor exponents

We can provably (and practically) compute $n_{C,p,\text{wild}}$ in the following cases:

- C an elliptic curve.
- C hyperelliptic, $p > 2$ or low genus.
- C superelliptic, p prime to exponent.
- C non-hyperelliptic of genus $3, 4$ or 5, $p > 2$.
- C plane quartic with a rational point.

Theorem (M^{2}D^2)

Let $C/\mathbb{Q} : y^2 = f(x)$ be a hyperelliptic curve, $p > 2$:

$$n_{C,p,\text{wild}} = \sum_{r \in R/G_{\mathbb{Q}_p}} v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p},$$

where R is the set of roots of f over $\bar{\mathbb{Q}}_p$.

Idea: generalise this formula to degree d covers of $\mathbb{P}^1$, $p > d$.

Toy example

Consider an elliptic curve $E/\mathbb{Q} : y^2 = x^3 + ax + b$ with $a \neq 0$. We consider the degree 3 cover to $\mathbb{P}^1$ and fill in the Galois diagram:

$$\begin{tikzcd}
& \mathbb{Q}(y,x,\Delta) = \mathbb{Q}(B) \arrow[dl, dash, "2"'] \arrow[ddr, dash, "3"] & \\
\mathbb{Q}(y,x) = \mathbb{Q}(E) \arrow[ddr, dash, "3"'] & & \\
& & \mathbb{Q}(y,\Delta) = \mathbb{Q}(D) \arrow[dl, dash, "2"] \\
& \mathbb{Q}(y) &
\end{tikzcd}$$

where $\Delta^2 = \mathrm{disc}_x(x^3 + ax + (b - y^2))$. It turns out $E[3] \cong \mathrm{Jac}(D)[3]$ and E and D have the same wild conductor exponents at $p \neq 3$.

Replacing $E \to \mathbb{P}^1$ by a degree 3 simply branched cover $C \to \mathbb{P}^1$, $\mathrm{Jac}(D)[3] \cong \mathrm{Jac}(C)[3] \oplus$ stuff as wild inertia reps for $p \neq 3$.

Main theorem

Theorem

Let $C \to \mathbb{P}^1$ be a degree d simply branched cover. For $p > d$,

$$n_{C,p,\text{wild}} = \sum_{r \in R/G_{\mathbb{Q}_p}} v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p},$$

where R is the set of $\bar{\mathbb{Q}}_p$-branch points.

Idea of proof

Unramified cyclic covers + Galois theory + representation theory

Perturbations

For $g \in \mathbb{Q}_p[t]$, write

$$w_p(g) = \sum_{r \in R/G_{\mathbb{Q}_p}} m(r) \cdot (v_p(\Delta(\mathbb{Q}_p(r)/\mathbb{Q}_p)) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p}),$$

where R is the set of $\bar{\mathbb{Q}}_p$-roots of g and r is a root with multiplicity $m(r)$.

Lemma

For p^{th}-power free g, the quantity $w_p(g)$ is locally constant.

Likewise, wild conductor exponents are locally constant. If we can perturb a degree d cover $\pi : C \to \mathbb{P}^1$ to obtain $\tilde{C} \to \mathbb{P}^1$ simply branched, then we can read off wild conductor exponents from the branch locus of π.

Example

Suppose $C : f(x, y) = 0$ is a smooth affine model for a curve and $\deg_x f = d$. Then, for $p > d$, we have $n_{C,p,\text{wild}} = w_p(\text{disc}_x f)$.