Stark's Conjectures and the eTNC Formalism

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Abstract

We introduce Stark's Main Conjecture on the leading coefficient of Artin L-functions at $s = 0$, and some of its natural refinements in the abelian case, before discussing how all of these refinements may be unified as a special case of the equivariant Tamagawa Number Conjecture (eTNC). We finish by showing how the eTNC can be used to generate further refinements.

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0 Introduction

We begin with a quick recall of the analytic class number formula (ACNF).

Definition 0.1. Given a number field k and a finite set of primes S of k including the infinite primes, we define the *S-truncated zeta function*

$$\zeta_{k,S}(s) = \prod_{\mathfrak{p} \notin S} (1 - N(\mathfrak{p})^{-s})^{-1}.$$

In the case where $S = S_\infty$ is the set of infinite primes, we have the celebrated analytic class number formula. This is typically stated as a recipe for the residue of the Dedekind zeta function $\zeta_k = \zeta_{k,S_\infty}$ at $s = 1$, but using the functional equation this is equivalent to the following [Tat84, Corollary 1.2].

Theorem 0.2 (Dedekind)**.** *Let k be a number field with r real embeddings and $2s$ complex embeddings. Denote the regulator of K by $R = R(k)$, the class number of k by $h = h(k)$, and the number of roots of unity in k by $w = w(k)$. Then ζ_k has a zero of order $r + s - 1 = |S_\infty| - 1$ at $s = 0$, and the leading coefficient of the Taylor expansion is*

$$-\frac{hR}{w}.$$

Given k and S we define h_S to be the class number of the ring

$$\mathcal{O}_S = \bigcap_{\mathfrak{p} \notin S} \mathcal{O}_{K,\mathfrak{p}},$$

where $\mathcal{O}_{K,\mathfrak{p}}$ is the completion of $\mathcal{O}_K$ at $\mathfrak{p}$. Denote by R_S the 'S-regulator' of K, given by the formula, where $\{u_i\}$ is a generating set for the torsion-free part of $\mathcal{O}_S^\times$ and $\mathfrak{p}_0$ is any choice of prime in S:

$$R_S = |\det(\log |u_i|_\mathfrak{p})_{\mathfrak{p} \in S - \mathfrak{p}_0}|. \tag{0.1}$$

Then the above theorem easily generalises to $\zeta_{k,S}$ in the following way.

Corollary 0.3. *Let k be a number field, and $S_\infty \subseteq S$ a set of primes. Then $\zeta_{k,S}$ has a zero of order $|S| - 1$ at $s = 0$ and the leading coefficient of the Taylor expansion is*

$$-\frac{h_S R_S}{w}.$$

Stark's conjectures are an attempt to weakly generalise the ACNF. In particular, we take K/k a Galois extension of number fields, with $G = \mathrm{Gal}(K/k)$, and S a finite set of primes of k including the infinite primes. We consider a complex representation V of G affording character χ. Then Stark's main conjecture [Tat84, Conjecture 5.1] predicts a recipe for a 'Stark regulator' such that the coefficient of the S-truncated Artin L-function of V at $s = 0$ (equivalently, $s = 1$) is a product of this regulator and an algebraic number.

Following Stark's work, there was for a time in the 1970s and '80s something of an industry dedicated to producing refinements of Stark's conjecture. We will discuss some of these refinements in this document, before turning our attention to the equivariant Tamagawa Number Conjecture (eTNC) of Bloch–Kato and Burns–Flach. This is hoped to be, in some sense, a universal refinement of Stark's conjecture.

In section 1 we explain Stark's main conjecture, and give some natural refinements in the case that K/k is an abelian extension, mostly following [Tat84]. In section 2 we give a special case of the eTNC and explain how this implies Stark's conjecture. In section 3 we show how the eTNC can be used to formulate refinements of known conjectures, following [Bur11a]. We also note that major progress has been made in recent years on some of the conjectural results discussed here (e.g. [DK23], [BBDS21]), and that this progress is, or may then be viewed as, evidence for the validity of the eTNC in general.

1 Stark's conjectures and refinements

Let K/k be a Galois extension of number fields, S a finite set of primes of k containing the infinite primes, and $S' = S(K)$ the set of primes of K lying above those in S. We consider a complex representation V of G affording character χ, and consider the corresponding S-truncated Artin L-function

$$L_S(\chi, s) = \prod_{\mathfrak{p} \notin S} \det(1 - \mathrm{Frob}_{\mathfrak{p}} N_{k/\mathbb{Q}}(\mathfrak{p})^{-s} | V^{I_{\mathfrak{p}}})^{-1},$$

where $I_{\mathfrak{p}}$ is the inertia subgroup at $\mathfrak{p}$ and $\mathrm{Frob}_{\mathfrak{p}}$ denotes the arithmetic Frobenius. From the functional equation of $L(\chi, s)$ we find the following:

Proposition 1.1. *$L_S(\chi, s)$ has a zero of order $r_S(\chi) = -\dim(V^G) + \sum_{\mathfrak{p} \in S} \dim(V^{G_{\mathfrak{p}}})$ at $s = 0$. Moreover,*

$$r_S(\chi) = \begin{cases} |S| - 1 & \textit{if } \chi = 1 \\ |\{\mathfrak{p} \in S | \chi(G_{\mathfrak{p}})\}|, & \textit{else.} \end{cases}$$

Notation. We denote denote the leading coefficient of the Taylor expansion at $s = 0$ by $L(\chi)$.

Then Stark's main conjecture, roughly-speaking, gives a recipe for a regulator $R(\chi)$, which generalises that introduced in the analytic class number formula. In particular, such that $L(\chi)/R(\chi)$ should be algebraic.

1.1 Stark's main conjecture

Fixing the notation of above, we now give the recipe for the Stark regulators we described. Although denoted above by $R(\chi)$, Stark regulators will also have an additional dependence on a choice of function f which we now describe.

Let X_S be the quotient of the free abelian group on S' given by

$$X_S = \left\{ \sum_{\mathfrak{P} \in S'} n_{\mathfrak{P}} \mathfrak{P} \ \middle| \ \sum_{\mathfrak{P} \in S'} n_{\mathfrak{P}} = 0 \right\}.$$

We also consider the set of S'-units $U_S = \{u \in K \mid ||u||_{\mathfrak{P}} = 1 \text{ for all } \mathfrak{P} \notin S'\}$. We write $X_{k,S}$ and $U_{k,S}$ for the analogous quotient of the free abelian group on S and the set of S-units respectively. Define a $\mathbb{C}$-linear map $\lambda_S : \mathbb{C}U_S \to \mathbb{C}X_S$[1]

$$\lambda_S : 1 \otimes u \mapsto \sum_{\mathfrak{P} \in S'} \log ||u||_{\mathfrak{P}} \mathfrak{P}.$$

Theorem 1.2 (Dirichlet's S-unit theorem)**.** *λ_S is an isomorphism.*

In fact, both $\mathbb{C}U_S$ and $\mathbb{C}X_S$ inherit actions of G from U_S and X_S and it is clear that λ_S is then an isomorphism of $\mathbb{C}[G]$-modules.

Definition 1.3 (Stark regulators)**.** Given any $\mathbb{C}[G]$-homomorphism $f : \mathbb{C}X_S \to \mathbb{C}U_S$, define the *Stark regulator* to be

$$R(\chi, f) = \det(\lambda_S \circ f \mid V),$$

where this denotes the determinant of the induced automorphism

$$\mathrm{Hom}_G(V^*, \mathbb{C}X_S) \to \mathrm{Hom}_G(V^*, \mathbb{C}X_S)$$

given by postcomposition with $\lambda_S \circ f$.

[1] We adopt the convention that for F a field and M a $\mathbb{Z}$-module (later, $\mathbb{Z}'$-module), the tensor product $F \otimes_{\mathbb{Z}} M$ is abbreviated to FM.

We borrow the following example of [Das99].

Example 1.4. Let us consider the link between this definition and the S-regulator defined in the introduction, $R_S = |\det(\log|u_i|_{\mathfrak{p}})_{\mathfrak{p}\in S-\mathfrak{p}_0}|$. We have G trivial, and so $\det(\lambda_S \circ f \mid V) = \det(\lambda_S \circ f)$. Moreover, we need only consider the trivial character. Write $R(f)$ for the Stark regulator in this case. Now take f to be injective. We have, for some fixed $\mathfrak{p}_0$

$$X_S = \bigoplus_{\mathfrak{p}\neq\mathfrak{p}_0} \mathbb{Z}(\mathfrak{p} - \mathfrak{p}_0).$$

Then with respect to the basis $\{\mathfrak{p} - \mathfrak{p}_0\}$, the matrix for $\lambda_S \circ f$ is $(\log|f(\mathfrak{p}_i - \mathfrak{p}_0)|_{\mathfrak{p}_j})_{i,j\neq 0}$ for some enumeration of S. Hence we find $R(f) = \det(\lambda_S \circ f) = \pm R_S[U_S : f(X_S)\mu(k)] = \pm R_S[U_S : f(X)]/w$, and the link to the analytic class number formula is clear with this set-up. $\diamondsuit$

It is an observation of Herbrand that we may take f to be an isomorphism defined on the level of $\mathbb{Q}[G]$-modules. Taking this to be the case, we are able to state Stark's main conjecture.

Conjecture 1.5 (Stark's Main Conjecture)**.** *Set $A(\chi, f) = R(\chi, f)/L(\chi)$. Then $A(\chi, f) \in \mathbb{Q}(\chi)$, and for all $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$*

$$A(\chi, f)^\sigma = A(\chi^\sigma, f).$$

Before we specialise to the abelian setting, we note some important properties of the quantity $A(\chi, f)$. Firstly, fixing f, $A(\chi, f)$ inherits the following 'Artin formalism' from $L(\chi, s)$:

Lemma 1.6. *(i) For all characters χ, χ' of G*

$$A(\chi + \chi', f) = A(\chi, f)A(\chi', f)$$

(ii) For $H \leq G$,

$$A(\mathrm{Ind}_H^G \chi, f) = A(\chi, f).$$

In fact this lemma is enough to deduce conjecture 1.5 for permutation representations. We also state some other cases where the conjecture is known to hold.

Theorem 1.7. *(i) Conjecture 1.5 holds for permutation representations.*
(ii) Conjecture 1.5 holds for all rational characters.
(iii) Conjecture 1.5 holds when $r(\chi) = 0$.

Sketch of proof. We sketch a proof of (i), and give some short discussion on the validity of (ii), (iii).

For (i), one can invoke the class number formula in the form of corollary 0.3 to show that the conjecture holds for the trivial character $\chi = 1_G$, then the conjecture holds for all characters which are sums of characters of the form $\mathrm{Ind}_H^G 1_H$.

We omit the proof of (ii), although we do note that (i) implies the weaker result that given a rational character χ, there exists a positive integer n such that conjecture 1.5 holds for $n\chi$. This is because there exists permutation characters such that $n\chi$ is their difference.

Lastly, (iii) follows from the non-trivial fact that $L(\chi^\sigma, 0) = L(\chi, 0)^\sigma$ for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, noting that $R(\chi, f) = 1$. $\square$

We note (iii) of the above as motivation for our consideration of the case $r(\chi) = 1$ in the following. For a full explanation of the results briefly described above, see [Tat84].

1.2 The rank one abelian case

We now maintain the same notation as above, but take K/k to be an abelian extension and consider $r(\chi) = 1$, so $L'_S(\chi, 0)$ is the leading term. We define, for non-empty $T \subset S$ containing places ramifiying in K/k,

$$U_{S,T} = \begin{cases} \{u \in U_S \mid ||u||_{\mathfrak{P}} = 1 \text{ for all } \mathfrak{P} \text{ lying above } T\}, & |T| \geq 2 \\ \{u \in U_S \mid ||u||_{\mathfrak{P}} \text{ is constant for } \mathfrak{P}|\mathfrak{p}\}, & T = \{\mathfrak{p}\} \end{cases}.$$

In this case we can then formulate a version of Stark's conjecture which predicts the existence of a *Stark unit*, with which we can give an explicit expression for the leading coefficient of $L_S(\chi, s)$ at $s = 0$:

Conjecture 1.8. *Suppose $|S| \geq 2$, and that $\mathfrak{p} \in S$ splits in K with $\mathfrak{P}|\mathfrak{p}$. Taking $T = S - \{\mathfrak{p}\}$, there exists $\varepsilon \in U_{S,T}$ such that*

$$L'_S(\chi, 0) = -\frac{1}{W} \sum_{\sigma \in G} \chi(\sigma) \log ||\varepsilon^\sigma||_{\mathfrak{P}},$$

and $K(\varepsilon^{1/W})/k$ is an abelian extension.

Remark 1.9. We will see in the following subsection that this is stronger than Stark's main conjecture in this setting. ♢

Notation. Following Tate, we denote the statement that such an ε exists for a specific extension K/k and set of primes S by $St(K/k, S)$. Note that we have omitted the choice of $\mathfrak{p}, \mathfrak{P}$ from this notation because $\varepsilon(\mathfrak{P}^\sigma) = \varepsilon(\mathfrak{P})^\sigma$ and because of the following.

Proposition 1.10. *$St(K/k, S)$ is true if S contains two places which split completely in K.*

This leads to the following simple corollaries.

Corollary 1.11. *(i) $St(K/k, S)$ holds if $K = k$.*
(ii) $St(K/k, S)$ holds if k has more than one complex place.

Proof. Trivially, every prime splits totally in K/k. All complex places split totally. □

1.3 The Brumer–Stark conjecture

We maintain the same setting as in the previous. In fact, we begin by introducing some definitions so that we can reformulate conjecture 1.8 in a way that will turn out to be more helpful.

Definition 1.12. For $\sigma \in G$, the *partial zeta function* $\zeta_{k,S}(\sigma, s) = \zeta_S(\sigma, s)$ is given by

$$\zeta_S(\sigma, k) = \sum_{\mathfrak{a} \triangleleft \mathcal{O}_k, (\mathfrak{a}, S)=1, \sigma_{\mathfrak{a}}=\sigma} N_{k/\mathbb{Q}}(\mathfrak{a})^{-s},$$

where $\sigma_{\mathfrak{a}}$ is the image of $\mathfrak{a}$ under the Artin reciprocity map.

This leads us to define

$$\theta_S(s) = \sum_{\sigma \in G} \zeta_S(\sigma, s)\sigma^{-1},$$

and to formulate

Conjecture 1.13. *Suppose $|S| \geq 2$, and that $\mathfrak{p} \in S$ splits in K with $\mathfrak{P}|\mathfrak{p}$. Taking $T = S - \{\mathfrak{p}\}$, there exists $u \in \mathbb{Q}U_S$ such that*

$$\lambda_S(u) = \begin{cases} -\theta'_S(0)\mathfrak{P}, & |T| \geq 2 \\ -\theta'_S(0)(\mathfrak{P} - \frac{1}{|G|}\mathfrak{q}), & T = \{\mathfrak{q}\} \end{cases}.$$

Moreover, there exists $\varepsilon \in K^\times$ such that the equality $Wu = 1 \otimes \varepsilon$ holds in $\mathbb{Q}K^\times$ and $K(\varepsilon^{1/W})/k$ is abelian.

Notation. Let us denote this statement for a specific choice of K/k and S by $St'(K/k, S)$, again suppressing the choice of $\mathfrak{p}$ and $\mathfrak{P}$.

Proposition 1.14. *$St'(K/k, S)$ and $St(K/k, S)$ are equivalent.*

Proof. We treat the case $|T| \geq 2$. The other case may be dealt with similarly.

Assume that $St(K/k, S)$ holds. Then take $u = 1/W \otimes \varepsilon$. Certainly the latter part of $St'(K/k, S)$ is valid. For the first part, we note that

$$L_S(\chi, s) = \sum_{\sigma \in G} \zeta_S(\sigma, s)\chi(\sigma)$$

and so the equality of $St(K/k, S)$ becomes

$$\sum_{\sigma \in G} \zeta'_S(\sigma, 0)\chi(\sigma) = -\frac{1}{W} \sum_{\sigma \in G} \chi(\sigma) \log ||\varepsilon^\sigma||_{\mathfrak{P}},$$

and we conclude $\log ||\varepsilon^\sigma||_{\mathfrak{P}} = -W\zeta'_S(\sigma, 0)$ and that

$$\lambda_S(u) = \frac{1}{W} \otimes \sum_{\mathfrak{Q} \in S'} \log ||\varepsilon||_{\mathfrak{Q}}\mathfrak{Q} = \frac{1}{W} \otimes \sum_{\mathfrak{Q}|\mathfrak{p}} \log ||\varepsilon||_{\mathfrak{Q}}\mathfrak{Q} = \frac{1}{W} \otimes \sum_{\sigma \in G} \log ||\varepsilon^\sigma||_{\mathfrak{P}} \times \sigma(\mathfrak{P}),$$

because $\varepsilon \in U_{S,T}$. Combining these facts yields $\lambda_S(u) = -\theta'_S(0)\mathfrak{P}$, and the converse works in exactly the same way. □

Notation. From now on we denote both $St'(K/k, S)$ and $St(K/K, s)$ by simply $St(K/k, S)$.

Remark 1.15. Conjecture 1.13 is stronger than Stark's main conjecture in this setting, in the sense that Stark's conjecture holding for all χ with $r(\chi) = 1$ is equivalent to conjecture 1.13 without the existence of ε. ♢

Now to formulate Brumer–Stark, we adopt a subtle change in perspective. We fix, instead of S, a non-empty set T of primes of k containing the infinite primes, and write

$$K^T = \begin{cases} \{u \in K \mid ||u||_{\mathfrak{Q}} = 1 \text{ for all } \mathfrak{Q} \text{ lying above } T\}, & |T| \geq 2 \\ \{u \in K \mid ||u||_{\mathfrak{Q}} \text{ is constant for } \mathfrak{Q}|\mathfrak{q}\}, & T = \{\mathfrak{q}\} \end{cases}.$$

We choose a prime $\mathfrak{p} \notin T$ which is totally split in K with $\mathfrak{P}|\mathfrak{p}$ and set $S = T \cup \{\mathfrak{p}\}$. It is a highly non-trivial fact that $\theta_T(0) \in \mathbb{Q}[G]$ (indeed Deligne and Ribet showed that $u\theta_T(0) \in \mathbb{Z}[G]$ for each $u \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$). Then we define the subgroup I_K^T of the ideal group I_K by

$$I_K^T = \{\mathfrak{I} \in I_K \mid \mathfrak{I}^{\theta_T(0)} = (u) \text{ such that } \exists\, \varepsilon \text{ with } Wu = \varepsilon \text{ in } \mathbb{Q}K^\times \text{ and } K(\varepsilon^{1/W})/K \text{ is abelian}\}.$$

Then we have

Conjecture 1.16 (Brumer–Stark). $I_K^T = I_K$.

Notation. Following Tate, we denote the statement that $I_K^T = I_K$ for a specific extension K/k and set of primes T by $BS(K/k,T)$.

The link to Stark's conjecture comes from the observation that the condition on u in conjecture 1.13 is equivalent to $u \in \mathbb{Q}K^T$ and $(u) = \theta_T(0)\cdot\mathfrak{P}$, where (u) is the image in $\mathbb{Q}I_K$ under the natural extension of the map $K^\times \to I_K$ given by $x \mapsto (x)$. This leads us to the following proposition.

Proposition 1.17. *Let P be a set of primes of k such that each $\mathfrak{p} \in P$ totally split in K and such that the primes of K lying above those in P generate the ideal class group Cl_K. Then*

$$BS(K/k,T) \iff St(K/k, T\cup\mathfrak{p}) \text{ for all } \mathfrak{p} \in P.$$

In particular, we may take P to be the set of all totally split primes.

Remark 1.18. We finish this subsection by noting that recent major progress has been made on the Brumer–Stark conjecture in [DK23]. In particular, if we rephrase conjecture 1.16 as the statement that $\theta_T(0)$ belongs to an appropriate annihilator A in the class group Cl_K (c.f. conjecture 3.1), then they prove this conjecture 'away from 2', meaning that $\theta_T(0)$ lies in $A \otimes_{\mathbb{Z}} \mathbb{Z}[1/2]$. $\diamondsuit$

1.4 Gross' refined class number formula

Lastly we discuss a conjecture of Gross; the 'refined class number formula', formulated in [Gro88]. As Stark's conjecture was an attempt to 'weakly generalise' the analytic class number formula, this conjecture is an attempt to obtain a more precise constraint on the relevant leading term. It gives a conjectural congruence satisfied by $\theta_S(0)$ modulo some power of the augmentation ideal $\ker(\mathbb{Z}[G] \to \mathbb{Z})$. For the sake of narrative ease, we state a slightly weakened special case of Gross' conjecture which essentially amounts to the case $T = \emptyset$ of the full '(S,T)-version' formulated in [loc. cit.]. In particular, recall from the preceding that $w\theta_S(0) \in \mathbb{Z}[G]$. To avoid certain techincalities, we set $\mathbb{Z}' = \mathbb{Z}[1/w]$ so that this statement becomes $\theta_S(0) \in \mathbb{Z}'[G]$. This will allow us to state a weakened version of Gross' conjecture that gives a congruence modulo the augmentation ideal $I = \ker(\mathbb{Z}'[G] \to \mathbb{Z}')$. We begin with the following preliminary fact:

Fact 1.19. *Suppose $\theta_S(0)$ vanishes to order r. We have $\theta_S(0) \in I^r$.*

Gross' refined class number formula will then be a conjectural congruence satisfied by $\theta_S(0)$, which we state as an equality in the graded piece I^r/I^{r+1}. To state this congruence, we will have to give a suitable notion of regulator in I^r/I^{r+1}. To do so, we first choose enumerations $\{\mathfrak{p}_0,\dots,\mathfrak{p}_r\}$ and $\{u_1,\dots,u_r\}$ of S and of $U'_{k,S}$ respectively. For each $\mathfrak{p}_i$, choose a place $\mathfrak{P}_i|\mathfrak{p}_i$ of K and let rec_i be the reciprocity map of local class field theory for the extension $K_{\mathfrak{P}_i}/k_{\mathfrak{p}_i}$. For each i, we define a homomorphism

$$\rho_i : k^\times \hookrightarrow k_{\mathfrak{p}_i}^\times \xrightarrow{\mathrm{rec}_i} G \to G' = G \otimes_{\mathbb{Z}} \mathbb{Z}'.$$

For later relevance, we view this as a pairing

$$\rho : U'_{k,S} \times X'_{k,S} \to G',$$

and define the regulator $\widetilde{R}_S \in I^r/I^{r+1}$ to be the image under the quotient map of

$$\det(\rho(u_j,\mathfrak{p}_i) - 1)_{1\le i,j\le r} \in I^r/I^{r+1},$$

via the identification $G' \xrightarrow{\sim} I/I^2$ given by $g \mapsto g-1$.

Remark 1.20. Recall that when defining Dirichlet's regulator (0.1), we took it to be positive. We lose this sign ambiguity here because $S_\infty \subset S$ implies that $2\widetilde{R}_S = 0$ (see the remarks following [Dar91, Conjecture 1.2.4]). ♢

Finally we are able to state Gross' refined class number formula:

Conjecture 1.21 (Gross)**.** *Let $\widetilde{\theta}_S(0)$ be the image of $\theta_S(0)$ in I^r/I^{r+1}. We have*

$$\widetilde{\theta}_S(0) = -h_S \widetilde{R}_S.$$

From this formulation it is clear that this is an attempt to generalise Dirichlet's analytic class number formula; indeed the minus sign is not a necessary inclusion and, moreover, if $\theta_S(0)$ does not vanish to order $r+1$, then we should have h_S odd by the remark. In this case the conjecture reads

$$\widetilde{\theta}_S(0) = \widetilde{R}_S,$$

cf. theorem 3.9 and corollary 3.10. We formulate it as above both to better resemble the ACNF, and because it better resembles the analogous formulation in the case of function fields. Due to the remark above we cannot recover the sign in the ACNF, but this is no great impediment. Less trivially, because we have weakened Gross' conjecture by applying everywhere the functor $-\otimes_{\mathbb{Z}} \mathbb{Z}'$, we cannot recover the ACNF in full.

2 A special case of the eTNC

2.1 Technical background

We begin with a brief discussion on determinant modules. We give the simple definition in the case of a free R-module of rank r

$$[M]_R = \bigwedge^r M \cong R,$$

where this (very non-canonical) isomorphism is given by $\wedge_i m_i \mapsto 1$, for any choice of R-basis $\{m_i\}$ of M. We extend this definition to the case that R is a finite dimensional semisimple commutative algebra and M is any finitely generated R-module in the following way. Write $R = \prod_i F_i$ as a (finite) product of fields so that M decomposes as $M = \bigoplus_i M_i$ for each M_i a free F_i-module. We write

$$[M]_R = \bigoplus_i [M_i]_{F_i}.$$

It is clear that these definitions agree in the case that M is a free R-module. We also define

$$[M]_R^{-1} = \mathrm{Hom}_R([M]_R, R).$$

These constructions have the following properties:

1. $[0]_R = R$.

2. For a short exact sequence $\mathcal{E} : 0 \to M \to N \to P \to 0$ of finitely generated free R-modules we obtain a canonical isomorphism

   $$\iota(\mathcal{E}) : [N]_R \to [M]_R \otimes_R [P]_R$$

   given by $(\wedge_i m_i) \wedge (\wedge_j \hat{p}_j) \mapsto (\wedge_i m_i) \otimes (\wedge_j p_j)$ for any choice of bases $\{m_i\}, \{p_j\}$ of M, P respectively and with $\hat{p}_j$ a choice of preimage of p_j in N.

3. We have caconical isomorphisms, both of which we will denote by ev_M,

   $$[M]_R \otimes_R [M]_R^{-1} \to R, \quad [M]_R^{-1} \otimes_R [M]_R \to R$$

   by $m \otimes f \mapsto f(m)$, $f \otimes m \mapsto f(m)$ respectively.

4. Given an isomorphism $f : M \to N$ of finitely generated free R-modules, we obtain canonical isomorphisms both denoted by $t(f)$

   $$[M]_R \otimes_R [N]_R^{-1} \xrightarrow{[f]_R \otimes 1} [N]_R \otimes_R [N]_R^{-1} \xrightarrow{\mathrm{ev}_N} R,$$

   $$[M]_R^{-1} \otimes_R [N]_R \xrightarrow{[f]_R^{-1} \otimes 1} [N]_R^{-1} \otimes_R [N]_R \xrightarrow{\mathrm{ev}_N} R,$$

   where $[f]_R$ is the map induced by f and $[f]_R^{-1}$ is the map $[M]_R^{-1} \to [N]_R^{-1}$ given by precomposition with $[f^{-1}]_R$.

Notation. From now on, we abbreviate $[M]_R \otimes_R [N_R]$ to simply $[M]_R[N]_R$.

Remark 2.1. We are ignoring some non-trivial techincalities in this set-up. For an explanation see [Bur11a, Remark 1.1]. $\diamondsuit$

The modules we will be interested in throughout the remainder of this work will be over the group algebra $\mathbb{Z}[G]$ (and its extensions of scalars), for some finite abelian group G − indeed this will later be taken to be $\mathrm{Gal}(K/k)$ for a finite abelian extension of number fields. In this setting, the above extends to G-modules of finite projective dimension and we obtain analogues of all of the properties discussed above, although it will no longer be true that $[M]_{\mathbb{Z}[G]}$ is a rank one free $\mathbb{Z}[G]$-module. For details, see [Bur11a].

To demonstrate that these modules are worthy of their name, and to see how they might be related to conjecture 1.5, we relate this construction to the determinants of R-module homomorphisms in the following way.

Lemma 2.2. *Let $f : M \to N$ be an isomorphism of (for simplicity) finitely generated free R-modules of rank r, and fix R-bases $\{m_i\}$, $\{n_i\}$ of M, N respectively. Set $\underline{m} = \wedge_i m_i$ and $\underline{n}^*$ to be the map $\wedge_i n_i \mapsto 1$. If β_M and β_N be the maps $\underline{m} \mapsto 1$ and $\underline{n}^* \mapsto 1$ respectively, and Φ is the matrix of f with respect to the bases $\{m_i\}$, $\{n_i\}$, then the following diagram commutes.*

$$\begin{array}{ccc} [M]_R[N]_R^{-1} & \xrightarrow{t(f)} & R \\ {\scriptstyle \beta_M\otimes\beta_N}\big\downarrow & & \big\uparrow{\scriptstyle \times\det(\Phi)} \\ R\otimes R & \xrightarrow{id} & R \end{array}$$

Proof. After unravelling the definitions we are required to show $\mathrm{ev}_N \circ ([f]_R \otimes 1) : r\underline{m} \otimes s\underline{n}^* \mapsto rs\det(\Phi)$, hence it suffices to show that $\underline{m} \otimes \underline{n}^* \mapsto \det(\Phi)$. We have

$$\underline{m} \otimes \underline{n}^* \mapsto \wedge_i\phi(m_i) \otimes \underline{n}^* \mapsto \underline{n}^*(\wedge_i\phi(m_i)) = \underline{n}^*(\bigwedge_i \sum_j \Phi_{ij}n_j) = \det(\Phi),$$

because the wedge product is antisymmetric. □

2.2 Statement of the conjecture

Let us recall the set-up for the abelian case of Stark's conjecture. We have K/k a finite abelian extension of number fields, and S a finite set of primes in k containing the infinite primes. Write $G = \mathrm{Gal}(K/k)$.

Proposition 2.3. *Suppose that the class group $Cl(\mathcal{O}_S)$ is trivial. Then there exists an exact sequence of $\mathbb{Z}[G]$-modules*

$$\tau_S : 0 \to U_S \to E_0 \xrightarrow{d} E_1 \to X_S \to 0$$

such that E_0, E_1 are finitely generated of finite projective dimension.

Remark 2.4. Building on Tate's work (see, for example, [Tat84, Theorem 5.1]), Chinburg constructed this 2-extension using the semi-local and global canonical classes of class field theory. ♢

The following allows us to lose the requirement that the class group is trivial.

Theorem 2.5. *Suppose that S contains all the primes which ramify in K/k. Then there exists unique (up to unique isomorphism), finitely generated $\widetilde{X}_S$ such that there exists an exact sequence of $\mathbb{Z}[G]$-modules*

$$0 \to Cl(\mathcal{O}_S) \to \widetilde{X}_S \to X_S \to 0$$

and that there exists a canonical (class of) 2-extension(s)

$$\tau_S : 0 \to U_S \to E_0 \xrightarrow{d} E_1 \to \widetilde{X}_S \to 0$$

with E_0, E_1 finitely generated of finite projective dimension which agrees with that of proposition 2.3 when $Cl(\mathcal{O}_s)$ is trivial.

We are almost able to define the *determinant lattice* Ξ_S. To do so, we note that τ_S gives rise to

$$\mathcal{E}_1 : 0 \to \mathbb{Q}U_S \to \mathbb{Q}E_0 \to \mathbb{Q}d(E_0) \to 0$$
$$\mathcal{E}_2 : 0 \to \mathbb{Q}d(E_0) \to \mathbb{Q}E_1 \to \mathbb{Q}\widetilde{X}_S \to 0$$

from which we obtain a $\mathbb{Q}[G]$-module isomorphism

$$\iota : [\mathbb{Q}E_0]_{\mathbb{Q}[G]}[\mathbb{Q}E_1]^{-1}_{\mathbb{Q}[G]} \xrightarrow{\mathrm{ev}_{\mathbb{Q}d(E_0)}\circ(\iota(\mathcal{E}_1)\otimes\iota(\mathcal{E}_2))} [\mathbb{Q}U_S]_{\mathbb{Q}[G]}[\mathbb{Q}\widetilde{X}_S]^{-1}_{\mathbb{Q}[G]}.$$

Lastly we take the $\mathbb{R}[G]$-module isomorphism ξ_S to be

$$\xi_S : [\mathbb{R}E_0]_{\mathbb{R}[G]}[\mathbb{R}E_1]^{-1}_{\mathbb{R}[G]} \xrightarrow{\mathbb{R}\otimes\iota} [\mathbb{R}U_S]_{\mathbb{R}[G]}[\mathbb{R}\widetilde{X}_S]^{-1}_{\mathbb{R}[G]} \xrightarrow{t(\lambda_S)} \mathbb{R}[G].$$

Definition 2.6 (Determinant lattice)**.**

$$\Xi_S = \xi_S([E_0]_{\mathbb{Z}[G]}[E_1]^{-1}_{\mathbb{Z}[G]}).$$

This construction, as one would hope, is independent of the choice of τ_S up to equivalence in $\mathrm{Ext}^2_{\mathbb{Z}[G]}(X_S, U_S)$:

Proposition 2.7. *Ξ_S does not depend on the specific choice of E_0, E_1; only on the class given by theorem 2.5.*

Proof. See [Bur11a, Proposition 1.11] and [loc. cit., Remark 1.12]. □

Finally, the relevant special case of the eTNC is as follows.

Conjecture 2.8 (eTNC)**.**

$$\mathbb{Z}[G] \cdot \theta^*_S(0) = \Xi_S.$$

Remark 2.9. The difficulty in verifying this case of the eTNC in general largely boils down to computing the canonical class τ_S. Despite this difficulty, in [BBDS21], Bullach, Burns, Daoud and Seo use the results of [DK23] to show that what is known as the 'minus part' of this case of the eTNC holds. Moreover, conjecture 2.8 is known fully in the following cases:

1. $k = \mathbb{Q}$ (Burns, Greither, Flach);
2. K/k is quadratic (Kim).

There are also strong results in this direction due to Bley in the case that K/F is an abelian extension of an imaginary quadratic field, and k is any intermediate field. See [Bur11a, Remark 2.9] for details. ♢

2.3 Analytic class number formula and Stark's conjecture

We began this document by claiming that Stark's conjecture arose as an attempt to generalise the analytic class number formula, so if we are to claim that this special case is (in some sense) a universal refinement of the abelian case of Stark's conjecture, then we'd better verify the following.

Proposition 2.10. *Conjecture 2.8 implies the analytic class number formula up to sign.*

Proof. Take $K = k$, so G is trivial and so is $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\widetilde{X}_S, U_S)$. Then there is only one class of 2-extensions and we have $U_S \cong \mathbb{Z}^{|S|-1} \times \mu(k)$, and $\widetilde{X}_S \cong X_S \times \mathrm{Cl}(\mathcal{O}_S) \cong \mathbb{Z}^{|S|-1} \times \mathrm{Cl}(\mathcal{O}_S)$, so we may take

$$\tau_S : 0 \to U_S \to E_0 \xrightarrow{0} E_1 \to \widetilde{X}_S \to 0,$$

where $E_0 = \mathbb{Z}^{|S|-1} \times \mu(k)$ and $E_1 = \mathbb{Z}^{|S|-1} \times \mathrm{Cl}(\mathcal{O}_S)$. Then we must compute the image under ξ_S of $[E_0]_{\mathbb{Z}}[E_1]_{\mathbb{Z}}^{-1}$. We note that

$$[E_0]_{\mathbb{Z}} = [\mu(k)]_{\mathbb{Z}}[\mathbb{Z}^{|S|-1}]_{\mathbb{Z}} = \frac{1}{w}[\mathbb{Z}^{|S|-1}]_{\mathbb{Z}} \text{ and } [E_1]_{\mathbb{Z}} = [\mathrm{Cl}(\mathcal{O}_S)]_{\mathbb{Z}}[\mathbb{Z}^{|S|-1}]_{\mathbb{Z}} = \frac{1}{h_S}[\mathbb{Z}^{|S|-1}]_{\mathbb{Z}}.$$

Hence we have

$$\xi_S : \frac{h_S}{w}[\mathbb{Z}^{|S|-1}]_{\mathbb{Z}}[\mathbb{Z}^{|S|-1}]_{\mathbb{Z}}^{-1} \xrightarrow{\mathbb{R}\otimes\iota(\mathcal{E}_1)\iota(\mathcal{E}_2)} [\mathbb{R}U_S]_{\mathbb{R}}[\mathbb{R}\widetilde{X}_S]_{\mathbb{R}}^{-1} \xrightarrow{t(\lambda_S)} \mathbb{R}[G],$$

and so, recalling lemma 2.2, conjecture 2.8 says

$$\mathbb{Z} \cdot \theta_S^*(0) = \Xi_S = \frac{h_S \det(\lambda_S)}{w} \cdot \mathbb{Z},$$

and so the leading term of $\theta_S(s) = \zeta_S(s)$ at $s = 0$ is $\pm h_S \det(\lambda_S)/w$. □

Remark 2.11. It is clear that this is the best we can hope for, because the eTNC is sensitive to changes in sign, whilst the regulator appearing in the analytic class number formula is taken to be an absolute value, c.f. remark 1.20. ♢

To see that the eTNC implies Stark's main conjecture in the abelian setting, we fix a $\mathbb{Q}[G]$-module isomorphism $f : \mathbb{Q}U_S \to \mathbb{Q}X_S$ and consider the quantity

$$R(f) = \det_{\mathbb{R}[G]}(\lambda_S \circ f^{-1}) \in \mathbb{R}[G]^{\times}.$$

Notation. We have a decompostion of $\mathbb{R}[G]$ as a finite product of fields, because $\mathbb{R}[G]$ is finite-dimensional and semisimple. Here, $\det_{\mathbb{R}[G]}$ denotes the product of the determinants calculated over each factor in this decompostion.

Proposition 2.12. *Stark's main conjecture in the abelian setting is equivalent to the statement*

$$\theta_S^*(0)R(f)^{-1} \in \mathbb{Q}[G]. \tag{2.1}$$

Proof. For characters $\chi \in \hat{G}$, write $\widetilde{\chi}$ for the $\mathbb{C}$-linear homomorphism $\mathbb{C}[G] \to \mathbb{C}$ given by $g \mapsto \chi(g)$ for $g \in G$. We obtain an isomorphism of $\mathbb{C}$-algebras

$$\mathbb{C}[G] \to \prod_{\chi\in\hat{G}} \mathbb{C},$$

given by $x \mapsto (\widetilde{\chi}(x))_{\chi\in\hat{G}}$: it is clear that this is a $\mathbb{C}$-algebra homomorphism. For bijectivity, note that there are $|G|$ distinct characters of G so $x = \sum_g x_g g \mapsto (\sum_g \chi(g)x_g)_{\chi\in\hat{G}}$ yields $|G|$ independent linear relations on the x_g, hence determining them uniquely. Identifying $\mathbb{C}[G]$ with this product yields

$$\mathbb{Q}[G] = \{(y_\chi)_{\chi\in\hat{G}} \mid y_\chi^\sigma = y_{\chi^\sigma} \text{ for all } \chi \in \hat{G} \text{ and } \sigma \in \mathrm{Aut}(\mathbb{C})\},$$

and so (2.1) becomes the statement

$$\widetilde{\chi}(\theta_S^*(0)R(f)^{-1})^\sigma = \widetilde{\chi^\sigma}(\theta_S^*(0)R(f)^{-1}) \text{ for all } \sigma \in \mathrm{Aut}(\mathbb{C}).$$

Stark's conjecture follows from the equality

$$\widetilde{\chi}(\theta_S^*(0)R(f)^{-1}) = \frac{L(\chi)}{\det(\lambda_S^{-1} \circ f \mid V_\chi)} = A(\chi, f)^{-1},$$

which we see because $\widetilde{\chi}(\theta_S^*(0)) = L(\chi)$ and $\widetilde{\chi}(R(f)) = \det(\lambda_S \circ f^{-1} \mid V_\chi)$ is the χ-component of $R(f)$. $\square$

This leads us to

Corollary 2.13. *Conjecture 2.8 (eTNC) implies Stark's main conjecture in the abelian setting.*

Proof. We show that eTNC implies that $\theta_S^*(0)R(f)^{-1} \in \mathbb{Q}[G]$, so that we are done by proposition 2.12. This is just unwinding the definition of the determinant lattice.

Upon tensoring with $\mathbb{Q}$, conjecture 2.8 gives

$$\begin{aligned}
\mathbb{Q}[G] \cdot \theta_S^*(0) &= \xi_S([\mathbb{Q}E_0]_{\mathbb{Q}[G]}[\mathbb{Q}E_1]_{\mathbb{Q}[G]}^{-1}) \\
&= t(\lambda_S)([\mathbb{Q}U_S][\mathbb{Q}X_S]^{-1}) \\
&= \mathrm{ev}_{\mathbb{Q}X_S}([\lambda_S(\mathbb{Q}U_S)][\mathbb{Q}X_S]^{-1}) \\
&= \mathrm{ev}_{\mathbb{Q}X_S}([\lambda_S \circ f^{-1}(\mathbb{Q}X_S)][\mathbb{Q}X_S]^{-1}) \\
&= \mathrm{ev}_{\mathbb{Q}X_S}([\mathbb{Q}X_S][\mathbb{Q}X_S]^{-1}) \cdot \det_{\mathbb{R}[G]}(\lambda_S \circ f^{-1}) \\
&= \mathrm{ev}_{\mathbb{Q}X_S}([\mathbb{Q}X_S][\mathbb{Q}X_S]^{-1}) \cdot R(f) \\
&= \mathbb{Q}[G] \cdot R(f),
\end{aligned}$$

and the claim follows immediately. $\square$

Remark 2.14. In addition to Stark's main conjecture and the other results discussed in this document, this special case of the eTNC is known to imply many other refinements of Stark's conjecture (see [Bur11a, page 19] for a list of examples). $\diamondsuit$

3 Next steps

We now discuss how our special case of the eTNC can lead us to formulate refinements of the well-known conjectures we have discussed in section 1. In this way, even though a proof in full generality may be far off, the eTNC can lead us to improvements of known results and conjectures which may be more amenable to proof.

3.1 Brumer's conjecture

We saw in the preceding that the implication

$$\text{eTNC} \implies \text{Stark's (abelian) conjecture}$$

followed after tensoring with $\mathbb{Q}$. We stated earlier that the Brumer–Stark conjecture came from an 'integral' version of Stark's conjecture in the abelian setting, so it is natural for one to explore the link between this refinement and conjecture 2.8. We use this example to demonstrate how the quite opaque eTNC allows us to formulate clear (potential) refinements.

We briefly recap the set-up: K/k is a finite abelian extension of number fields with $G = \mathrm{Gal}(K/k)$, and S is a finite set of places of k including the infinite places and those which ramify in K.

We will actually consider the following weaker[2] version of the Brumer-Stark conjecture, due only to Brumer.

Conjecture 3.1 (Brumer)**.** *For each $x \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$ and $S_\infty \subseteq T \subseteq S$,*

$$x\theta_S(0) \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_T)).$$

Burns in [Bur11a] poses a question which suggests a generalisation of the above. Clearly Brumer's conjecture is not interesting if $\theta_S(0) = 0$, but recall that in this case the statement of Stark's (abelian) conjecture (proposition 2.12) says $\theta_S^*(0) \cdot R(f)^{-1} \in \mathbb{Q}[G]$. Hence, writing $\theta_S^{(r)}(s) = \theta_S(s)/s^r$, the following may be a reasonable generalisation:

Question 3.2 (Burns)**.** *Suppose $\theta_S(0)$ vanishes to order r. For each $x \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$ and $f \in \mathrm{Hom}_G(U_S, X_S)$, is it the case that*

$$x\theta_S^{(r)}(0) \cdot R(f)^{-1} \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mathrm{Cl}(\mathcal{O}_S))?$$

Remark 3.3. Macias Castillo shows in [MC13] that question 3.2 has a positive answer for K/k a quadratic extension, amongst some other cases. ♢

We are cautious to note that a positive answer to question 3.2 would not imply Stark's (abelian) conjecture because $\theta_S^*(0)$ is, in general, different from $\theta_S^{(r)}(0)$. However, we can obtain a link to a 'p-adic Stark conjecture' by observing that question 3.2 naturally splits into the following questions indexed by prime p:

Question 3.2$_\mathbf{p}$. *Suppose $\theta_S(0)$ vanishes to order r. For each $x \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$ and $f \in \mathrm{Hom}_G(U_S, X_S)$, is it the case that*

$$x\theta_S^{(r)}(0) \cdot R(f)^{-1} \in \mathrm{Ann}_{\mathbb{Z}[G]}(Cl(\mathcal{O}_S)) \otimes \mathbb{Z}_{(p)}?$$

[2]To be chronologically precise, one should point out that the Brumer–Stark conjecture is a generalisation of Brumer's conjecture. The former was stated by Tate, who combined Brumer's original ideas with those of Stark.

These questions have been studied by Burns and Macias Castillo, amongst others. The answer is known to be positive under each of the following circumstances:

1. There are r places in S which are totally split in K/k and $\mu(K)\otimes\mathbb{Z}_{(p)}$ has finite projective dimension;
2. $p\nmid[K:k]$.

We promised a link to a 'p-adic Stark conjecture'. Indeed, Burns shows the following (see [Bur11a, Corollary 3.15]):

Proposition 3.4. *Suppose that the 'p-adic Stark conjecture at $s=1$' holds for K/k. Then the answer to question 3.2_p is postive.*

3.2 Refined class number formulas

We saw in proposition 2.10 that the eTNC implies the ANCF up to sign, so it is natural to ask whether we can also deduce strengthenings of the ANCF such as Gross' refined class number formula. We discuss here how congruences such as Gross' conjecture can be formulated from conjecture 2.8. As in the earlier discussion of Gross' refined class number formula, we will avoid certain techincalities by applying everywhere the functor $-\otimes_{\mathbb{Z}}\mathbb{Z}'$. We begin with the following lemma.

Lemma 3.5. *There exists an exact sequence*

$$\tau'_S : 0\to U'_S\to F\xrightarrow{\phi}F\xrightarrow{\pi}\widetilde{X}'_S\to 0$$

such that F is a finitely generated free $\mathbb{Z}'[G]$-module and τ'_S corresponds to the class τ_S under the natural isomorphism $\mathrm{Ext}^2_{\mathbb{Z}[G]}(\widetilde{X}_S,U_S)\otimes_{\mathbb{Z}}\mathbb{Z}'\cong\mathrm{Ext}^2_{\mathbb{Z}'[G]}(\widetilde{X}'_S,U'_S)$.

Proof. See [Bur11a, Lemma 3.2]. □

This allows us to compute $\Xi'_S=\Xi_S\otimes_{\mathbb{Z}}\mathbb{Z}'$ by mimicking precisely the construction preceding definition 2.6, of the determinant lattice Ξ_S. The use of this is that it gives us the following (this is [Bur11a, Exercise 3.3]):

Lemma 3.6. *Choose sections ι_1 and ι_2 to the surjections $\mathbb{R}F\xrightarrow{\phi}\mathbb{R}\mathrm{im}(\phi)$ and $\mathbb{R}F\xrightarrow{\pi}\mathbb{R}\widetilde{X}'_S$ respectively. We have*

$$\Xi'_S=\mathbb{Z}'[G]\cdot\det_{\mathbb{R}[G]}(\hat{\phi}),$$

where $\hat{\phi}$ is the unique $\mathbb{R}[G]$-module automorphism of $\mathbb{R}F$ which agrees with $\iota_2\circ\lambda_S$ on U'_S (viewed as a submodule of F) and with ϕ on $\iota_1(\mathbb{R}\mathrm{im}(\phi))$.

Proof. First note that we have $\det_{\mathbb{R}[G]}(\hat{\phi})=\det_{\mathbb{R}[G]}(\iota_2\circ\lambda_S)\cdot\det_{\mathbb{R}[G]}(\phi\circ\iota_1)$, because we have $\mathbb{R}F\cong\mathbb{R}U'_S\oplus\mathbb{R}\mathrm{im}(\phi)$. Recall that lemma 2.2 gives us a commutative diagram

$$\begin{tikzcd}
{[\mathbb{R}U'_S]_{\mathbb{R}[G]}[\mathbb{R}U'_S]^{-1}_{\mathbb{R}[G]}} \arrow[r, "t(\iota_2\circ\lambda_S)"] \arrow[d] & \mathbb{R}[G] \\
\mathbb{R}[G]\otimes\mathbb{R}[G] \arrow[r, "\mathrm{id}"] & \mathbb{R}[G] \arrow[u, "\times\det_{\mathbb{R}[G]}(\iota_2\circ\lambda_S)\cdot"']
\end{tikzcd}$$

Likewise, the diagram

$$\begin{tikzcd}
{[\mathbb{R}\mathrm{im}(\phi)]_{\mathbb{R}[G]}[\mathbb{R}\mathrm{im}(\phi)]^{-1}_{\mathbb{R}[G]}} \arrow[r, "t(\phi\circ\iota_1)"] \arrow[d] & \mathbb{R}[G] \\
\mathbb{R}[G]\otimes\mathbb{R}[G] \arrow[r, "\mathrm{id}"] & \mathbb{R}[G] \arrow[u, "\times\det_{\mathbb{R}[G]}(\phi\circ\iota_1)"']
\end{tikzcd}$$

commutes. Unravelling the construction of Ξ'_S (as for Ξ_S), we have that this is the image of $[\mathbb{Q}F][\mathbb{Q}F]^{-1}$ under the map

$$[\mathbb{R}F]_{\mathbb{R}[G]}[\mathbb{R}F]^{-1}_{\mathbb{R}[G]} \to [\mathbb{R}U'_S]_{\mathbb{R}[G]}[\mathbb{R}\mathrm{im}(\phi)]_{\mathbb{R}[G]}[\mathbb{R}\mathrm{im}(\phi)]^{-1}_{\mathbb{R}[G]}[\mathbb{R}\widetilde{X}'_S]^{-1}_{\mathbb{R}[G]} \to [\mathbb{R}U'_S]_{\mathbb{R}[G]}[\mathbb{R}\widetilde{X}'_S]^{-1}_{\mathbb{R}[G]} \to \mathbb{R}[G],$$

and identifying $\widetilde{X}'_S$ with U'_S via ι_2, we see from above that the diagram

$$\begin{tikzcd}
{[\mathbb{R}F]_{\mathbb{R}[G]}[\mathbb{R}F]^{-1}_{\mathbb{R}[G]}} \arrow[r] \arrow[d] & \mathbb{R}[G] \\
\mathbb{R}[G] \otimes \mathbb{R}[G] \arrow[r, "\mathrm{id}"] & \mathbb{R}[G] \arrow[u, "\times \det_{\mathbb{R}[G]}(\phi\circ\iota_1)\cdot\det_{\mathbb{R}[G]}(\iota_2\circ\lambda_S)"']
\end{tikzcd}$$

commutes; hence the result. □

For each irreducible character $\chi \in \hat{G}$ of G, we consider the element

$$e_\chi = \frac{\chi(1)}{|G|}\sum_{\sigma\in G}\chi(\sigma^{-1})\sigma$$

of $\mathbb{C}[G]$. Then e_χ is a central idempotent and for an irreducible complex representation V of G we have

$$e_\chi(V) = \begin{cases} V, & \text{if } V \text{ affords character } \chi \\ 0, & \text{else.} \end{cases}$$

In particular, this allows us to rewrite $\theta_S(0) = \sum_{\chi\in\hat{G}} L(\overline{\chi}, 0)e_\chi$. Note that the complex conjugation here is essentially an artefact of our use of the arithmetic Frobenius in the definition of Artin L-functions, rather than the *geometric* Frobenius. We will also consider the element e_0, defined by

$$e_0 = \sum_{\chi\in\hat{G}_0} e_\chi,$$

where $\hat{G}_0 = \{\chi \in \hat{G} \mid L(\overline{\chi}, 0) = 0\}$. This is a natural element to introduce, as we intend to study $\theta^*_S(0) = \theta_S(0)e_0$. Note that we also have the alternative descriptions

$$\hat{G}_0 = \{\chi \in \hat{G} \mid e_\chi(\mathbb{C}X_S) = 0\} = \{\chi \in \hat{G} \mid e_\chi(\mathbb{C}U_S) = 0\}. \tag{3.1}$$

Then we have

Proposition 3.7. *The eTNC (conjecture 2.8) implies that there exists a unit $u \in \mathbb{Z}'[G]$ such that*

$$\theta_S(0) = u \cdot \det_{\mathbb{Z}'[G]}(\phi).$$

Proof. We have

$$\begin{aligned}
\theta_S(0) &= \theta^*_S(0)e_0 \\
&= u \cdot \det_{\mathbb{R}[G]}(\hat{\phi})e_0 && \text{(eTNC + lemma 3.6)} \\
&= u \cdot \det_{\mathbb{R}[G]}(\phi)e_0 && (e_0(\mathbb{Q}U_S) = 0 \text{ by (3.1)}) \\
&= u \cdot \det_{\mathbb{Z}'[G]}(\phi).
\end{aligned}$$

The final equality follows from the fact that $\det_{\mathbb{Z}'[G]}(\phi) = \det_{\mathbb{Z}'[G]}(\phi)e_0$, which in turn follows from the implication $\chi(\mathbb{C}\ker(\phi)) = 0 \implies \chi \in \hat{G}_0$. Indeed, suppose $\chi(\mathbb{C}\ker(\phi)) = 0$. Then upon identifying U'_S with $\ker(\phi)$, the claim follows from (3.1). □

We will combine proposition 3.7 with the following in order to obtain a congruence condition on $\theta_S(0)$.

Lemma 3.8. *There exists a $\mathbb{Z}'[G]$-basis $\{f_i\}_{1\le i\le d}$ for F such that*

1. $F_1 = \langle f_1, \dots, f_r\rangle_{\mathbb{Z}'[G]}$ *satisfies* $F_1^G = \ker(\phi^G)$;
2. $F_2 = \langle f_{r+1}, \dots, f_d\rangle_{\mathbb{Z}'[G]}$ *satisfies* $\phi(F_2^G) \subseteq F_2^G$.

Proof. See [Bur11a, Lemma 3.2]. □

This tells us that, with respect to the basis $\{f_i\}_{1\le i\le d}$ of F, the matrix for ϕ is a block matrix of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in M_{r\times r}(I)$, $B \in M_{(d-r)\times r}(I)$, $C \in M_{r\times(d-r)}(I)$, and $D \in M_{(d-r)\times(d-r)}(\mathbb{Z}'[G])$. Hence the equality of proposition 3.7 becomes

$$\theta_S(0) \equiv u \cdot \det D \det A \pmod{I^{r+1}}.$$

This congruence will turn out to be equivalent to the version of Gross' refined class number formula we stated earlier. To show this we consider the complexes $F \xrightarrow{\phi} F$ and $F^G \xrightarrow{\phi^G} F^G$, denoted $F^\bullet$ and $F^{\bullet,G}$ respectively. From the definition of I we obtain

$$0 \to I \otimes_{\mathbb{Z}'[G]} F^\bullet \to F^\bullet \to F^{\bullet,G} \to 0$$

and hence, via the snake lemma, a homomorphism $\ker(\phi)^G \to I \otimes_{\mathbb{Z}'[G]} \mathrm{cok}(\phi)$. Thereby we obtain a map $\ker(\phi)^G \to I/I^2 \otimes_{\mathbb{Z}'} \mathrm{cok}(\phi)_G$, and so a pairing

$$\ker(\phi)^G \times \mathrm{Hom}_{\mathbb{Z}'}(\mathrm{cok}(\phi)_G, \mathbb{Z}') \to I/I^2.$$

Recalling lemma 3.5, we identify $\ker(\phi)^G$ with $U'_{k,S}$ and $\mathrm{cok}(\phi)_G$ with $X'_{k,S}$. We also identify I/I^2 with $G' = G \otimes_{\mathbb{Z}} \mathbb{Z}'$ as before. Finally then, this gives us a pairing

$$\rho : U'_{k,S} \times X'_{k,S} \to G'.$$

We finish by noting that this pairing agrees with the pairing ρ defined in section 1.4.

Theorem 3.9. *The two pairings ρ defined here and in section 1.4 agree. There exist bases of $U'_{k,S}$ and $X'_{k,S}$ such that*

$$\det(\rho(u_i, x_j) - 1)_{1\le i,j\le r} = u \cdot \det(D)\det(A) \in I^r/I^{r+1}.$$

Proof. See [Bur11a, Lemma 3.5 & Theorem 3.7]. □

This shows

Corollary 3.10.

$$\text{eTNC} \implies \text{conjecture 1.21.}$$

We also note that only theorem 3.9 required specific knowledge of the construction of τ_S, and hence this methodology is widely applicable in other special cases of the eTNC to deduce congruences on leading terms of L-functions at critical values.

4 Epilogue

The goal of this project has been to introduce the reader—perhaps more accurately, the author—to the equivariant Tamagawa Number Conjecture via the more approachable conjecture of Stark and its refinements. In particular, the goal has been to appreciate that understanding a conjecture's place in the wider conjectural framework is important in studying the original conjecture and its refinements, even if one is not attempting to tackle the wider conjectures directly. Indeed, we began this document by discussing the analytic class number formula, which led to Stark's conjectures and their refinements, but we can also study other leading term (conjectural) results in similar ways. For example—and of most interest to the author—one might consider a 'cousin' of the ACNF: the conjectural leading term formula given by the Birch–Swinnerton-Dyer conjecture (BSD). A different special case of the eTNC can be formulated which is to BSD as conjecture 2.8 is to the ACNF (see [BC19]), by which we mean that the eTNC should lead in some sense to a 'universal refinement' of BSD. In fact, using essentially the same strategy as we used to deduce corollary 3.10, one may deduce analogous congruences for the leading terms of Hasse-Weil L-functions at $s = 1$.

The eTNC is thought in general to encompass leading term conjectures for the broad class of motivic L-functions. See, for example, [Ven07, §2] for an introduction to motives and motivic L-functions.

In this project we have largely kept ourselves to the case of abelian extensions of number fields, but we should point out that much of what has been said generalises naturally to the case of non-abelian extensions and to the case of global function fields. In the function field case (as is common) things turn out to be more simple; Burns shows in [Bur11b] that the analogue of conjecture 2.8 is known to hold for function fields.

As already discussed, we have seen how the eTNC can be used to generate further refinements of more simple well-known conjectures, but we have spoken little about the eTNC being a pathway to verifying these large classes of refinements. Before we end we should briefly discuss this path. Tackling conjecture 2.8 in general is extremely difficult and, as mentioned in remark 2.9, much of this difficulty boils down to computing the canonical class τ_S in general. More general cases of the eTNC are more difficult still, because it is no longer sufficient to work with the singular perfect 2-extension τ_S; rather one must work with 'perfect complexes' over $\mathbb{Z}_p[G]$ for each prime p and piece together this information over different primes.

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