Motivic pieces of curves

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6 September 2023

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Introduction

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Image: A matrix and a matrix

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This is the Weil conjectures in action.

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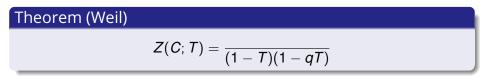
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Theorem (Weil)

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Theorem (Lefschetz trace formula)

$$\#\operatorname{Fix}(\alpha) = \sum_{k=0}^{2} (-1)^{k} \operatorname{Tr}(\alpha^{*} \mid H^{k}(\overline{C}, \mathbb{Q}_{\ell}))$$

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We'll do this for H^1 , which is the 'interesting' part of the cohomology of a curve.

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Let's look at the piece on which α acts by *z*; say this is the ρ -piece.

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We find that the characteristic polynomial of Φ acting on the ρ -piece is:

$$T^3 - 2(\zeta_3 - 1)T^2 + 10(\zeta_3 - 1)T + 125\zeta_3.$$

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and we computed the trace (hence characteristic polynomial) of Φ acting on here.

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In this way, we associate to each motive *M* an *L*-function L(M, s).

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Theorem

Analytic continuation, BSD, Galois equivariance of ranks (Deligne) imply

$$\operatorname{ord}_{s=1} L(C, \rho, s) = \langle \operatorname{Jac}(C)_{\mathbb{Q}(\zeta_3)}, \rho \rangle.$$

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The *L*-functions discussed before arise in this way when we allow Galois extensions of function fields not coming from extensions of the field of constants.

Thanks for listening!

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