# Motivic pieces of curves

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This is the Weil conjectures in action.

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## Theorem (Lefschetz trace formula)

$$\#\,\mathrm{Fix}(\alpha) = \sum_{k=0}^{2} (-1)^k\,\mathrm{Tr}(\alpha^* \mid H^k(\overline{C}, \mathbb{Q}_\ell))$$

# Curves with automorphisms

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We'll do this for $H^1$, which is the 'interesting' part of the cohomology of a curve.

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Let's look at the piece on which $\alpha$ acts by $z$; say this is the $\rho$-piece.

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We find that the characteristic polynomial of $\Phi$ acting on the $\rho$-piece is:

$$T^3 - 2(\zeta_3 - 1)T^2 + 10(\zeta_3 - 1)T + 125\zeta_3.$$

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and we computed the trace (hence characteristic polynomial) of $\Phi$ acting on here.

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In this way, we associate to each motive $M$ an $L$-function $L(M,s)$.

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**Theorem**

*Analytic continuation, BSD, Galois equivariance of ranks (Deligne) imply*

$$\mathrm{ord}_{s=1} L(C,\rho,s) = \langle \mathrm{Jac}(C)_{\mathbb{Q}(\zeta_3)}, \rho\rangle.$$

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The $L$-functions discussed before arise in this way when we allow Galois extensions of function fields not coming from extensions of the field of constants.

Thanks for listening!