# Wild conductor exponents of trigonal and tetragonal curves

Harry Spencer

University College, London

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**Remark**

The tame conductor exponent, $n_{C,p,\text{tame}}$, can be computed from a regular model of $C/\mathbb{Q}_p$.

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Today: $C : f(x, y) = 0$ smooth away from infinity with $\deg_x f \in \{3, 4\}$, $p > 3$.

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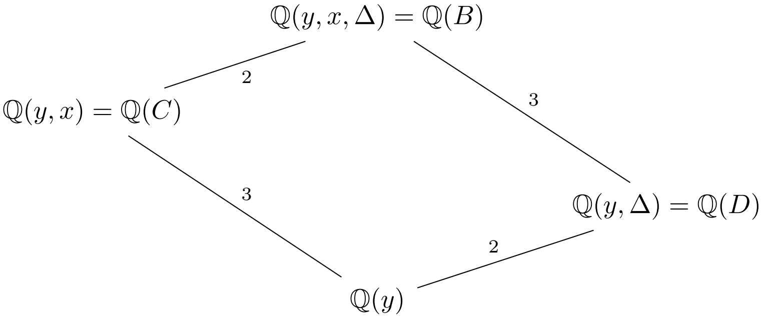

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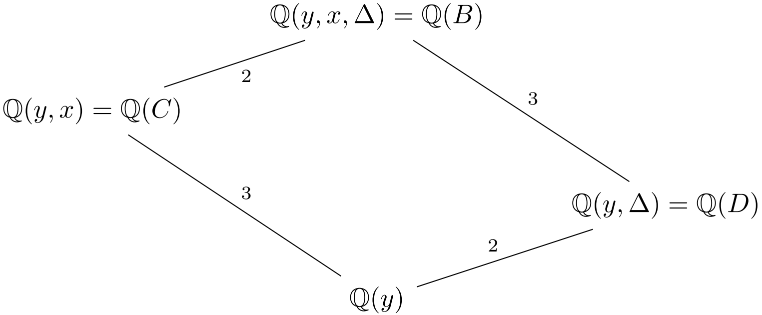


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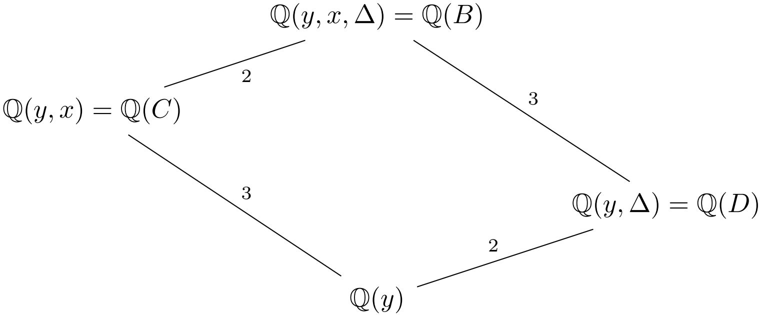
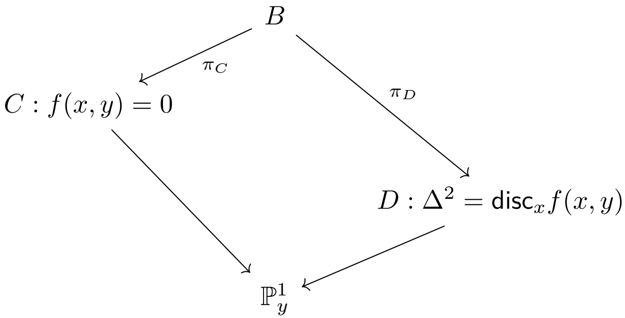



Where $\Delta^2 = \text{disc}_x f(x, y)$. Note that the genus of $D$ is at most $g(C) + 1$, and that this bound is sharp when the cover $C \to \mathbb{P}^1$ is simply ramified.

# A map on 3-torsion

Write $\tau$ for a generator of Gal$(\mathbb{Q}(y, x, \Delta)/\mathbb{Q}(y, \Delta))$.



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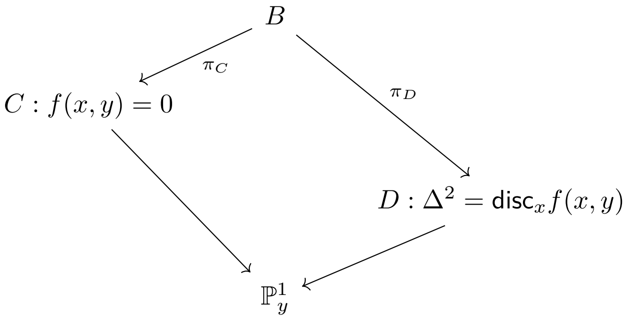


**Theorem**

*The map* $\pi_C^* - \tau \circ \pi_C^* : J_C[3] \to J_B[3]$ *is surjective (or injective) onto* $\pi_D^*(J_D[3])$.

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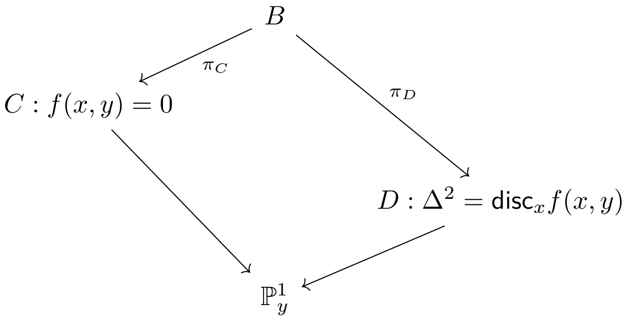


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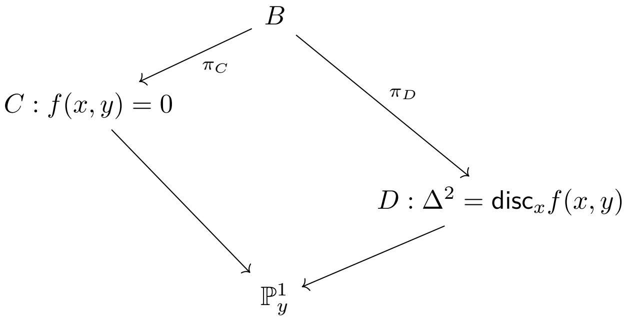


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This comes from studying the kernel of the isogeny

$$\phi : J_C \times J_C \times J_D \to J_B$$

given by $(P, Q, R) \mapsto \pi_C^*(P) + \tau(\pi_C^*(Q)) + \pi_D^*(R)$.

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**Theorem (Dokchitser²–Maistret–Morgan)**

*The wild conductor exponent at $p \neq 2$ of the hyperelliptic curve $H/\mathbb{Q} : s^2 = g(t)$, with $g$ square-free, is*

$$n_{H,p,\textit{wild}} = \sum_{r \in R/G_{\mathbb{Q}_p}} v_p(\Delta_{\mathbb{Q}_p(r)/\mathbb{Q}_p}) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p(r)/\mathbb{Q}_p},$$

*where $R$ is the set of roots of $g$ over $\bar{\mathbb{Q}}_p$.*

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$$n_{C,5,\mathsf{wild}} = 7 - 5 + 1 = 3.$$

# The degree 4 case

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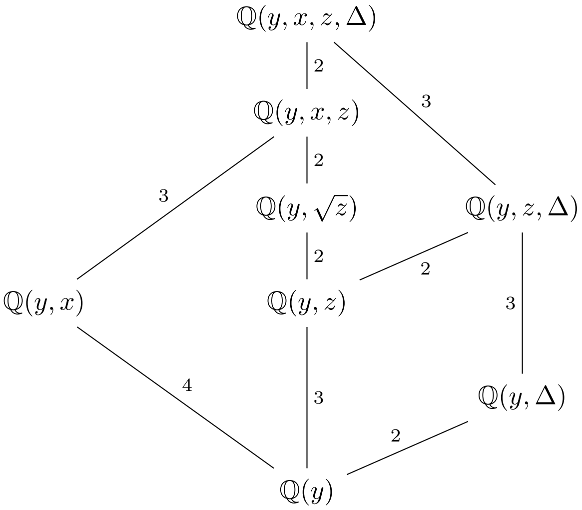
As before $D : \Delta^2 = \text{disc}_x f(x,y)$.

# The $S_4$-diagram



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which is (thankfully!) the same result as earlier.

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Suppose $C : f(x,y) = 0$ is smooth with $\deg_x f = m$.

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## Question

Suppose $C : f(x,y) = 0$ is smooth with $\deg_x f = m$. Do we have

$$n_{C,p,\text{wild}} = \sum_{r \in R/G_{\mathbb{Q}_p}} m(r) \cdot (v_p(\Delta_{\mathbb{Q}_p(r)/\mathbb{Q}_p}) - [\mathbb{Q}_p(r) : \mathbb{Q}_p] + f_{\mathbb{Q}_p Q(r)/\mathbb{Q}_p}),$$

for $R$ the roots of $\text{disc}_x f(x,y)$ if $p > m$?