Stark's Conjectures and the eTNC Formalism

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The full statement is difficult and opaque, so I will just give a special case which is closer to the ACNF side of things.

Structure

1. Analytic Class Number formula
2. Stark's Conjectures
3. The eTNC: Background
4. The eTNC: Statement
5. Stark's Conjectures and the eTNC
6. What next?

Analytic class number formula

First, we recall the definition of the Dedekind ζ-function for a number field k:

$$\zeta_k(s) = \prod_{\mathfrak{p} \notin S_\infty} (1 - N(\mathfrak{p})^{-s})^{-1},$$

where S_∞ is the set of infinite places.

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where S_∞ is the set of infinite places.

In fact, we will consider a slight modification:

Definition (*S-truncated ζ-function*)

$$\zeta_{k,S}(s) = \prod_{\mathfrak{p} \notin S} (1 - N(\mathfrak{p})^{-s})^{-1},$$

for a finite set of primes S containing S_∞.

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and, for $\{u_i\}$ generators for $\mathcal{O}_S^\times/\text{tors}$ and some choice $\mathfrak{p}_0 \in S$,

$$R_S = \left|\det(\log|u_i|_\mathfrak{p})_{\mathfrak{p}\in S-\mathfrak{p}_0}\right|.$$

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This is well-defined up to conjugation (and inertia).

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Definition (*S-truncated Artin L-function*)

For (χ, V) a representation of G,

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Stark predicts a recipe for a 'Stark regulator' such that the leading coefficient $L_S(\chi)$ of $L_S(\chi, s)$ at $s = 0$ is a product of this regulator and an algebraic number.

Stark's conjectures

We give the recipe for Stark's regulator. Define

$$X_S = \left\{ \sum_{\mathfrak{P} \in S'} n_{\mathfrak{P}} \mathfrak{P} \;\middle|\; \sum_{\mathfrak{P} \in S'} n_{\mathfrak{P}} = 0 \right\}$$

and

$$U_S = \{ u \in K \;|\; ||u||_{\mathfrak{P}} = 1 \text{ for all } \mathfrak{P} \notin S' \}.$$

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Theorem (*Dirichlet's S-unit theorem*)

The $\mathbb{C}$-linear map $\lambda_S : \mathbb{C}U_S \to \mathbb{C}X_S$ via

$$1 \otimes u \mapsto \sum_{\mathfrak{P} \in S'} \log ||u||_{\mathfrak{P}} \mathfrak{P}$$

is an isomorphism of $\mathbb{C}[G]$-modules.

Stark's conjectures

Given any $\mathbb{C}[G]$-homomorphism $f : \mathbb{C}X_S \to \mathbb{C}U_S$, define Stark's regulator

$$R(\chi, f) = \det(\lambda_S \circ f \mid V),$$

where this denotes the determinant of the induced automorphism

$$\mathrm{Hom}_G(V^*, \mathbb{C}X_S) \to \mathrm{Hom}_G(V^*, \mathbb{C}X_S)$$

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given by postcomposition with $\lambda_S \circ f$. Choose f to be a $\mathbb{Q}[G]$-isomorphism:

Conjecture (*Stark's Main Conjecture*)

Set $A(\chi, f) = R(\chi, f)/L(\chi)$. Then $A(\chi, f) \in \mathbb{Q}(\chi)$, and for all $\sigma \in Gal(\mathbb{Q}(\chi)/\mathbb{Q})$

$$A(\chi, f)^\sigma = A(\chi^\sigma, f).$$

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We give a brief introduction to *determinant modules*, restricting to the case of free R-modules M of finite rank r:

Definition

$$[M]_R = \bigwedge^r M \cong R \quad \text{and} \quad [M]_R^{-1} = \mathrm{Hom}_R([M]_R, R).$$

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This extends to finitely generated R-modules M for $R = \mathbb{Q}[G], \mathbb{C}[G]$, etc. for finite abelian groups G by writing

$$R = \prod_i F_i \quad \text{and} \quad M = \bigoplus_i M_i$$

for F_i fields and M_i a free F_i-module, and taking

$$[M]_R = \prod_i [M_i]_{F_i}.$$

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This construction has the following properties:

1. Given $\mathcal{E} : 0 \to M \to N \to P \to 0$, we obtain canonical

$$\iota(\mathcal{E}) : [N]_R \xrightarrow{\sim} [M]_R \otimes_R [P]_R.$$

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3. Given $f : M \xrightarrow{\sim} N$, we obtain canonical isomorphism

$$t(f) : [M]_R \otimes_R [N]_R^{-1} \xrightarrow{[f]_R \otimes 1} [N]_R \otimes_R [N]_R^{-1} \xrightarrow{\mathrm{ev}_N} R,$$

where $[f]_R$ is the map induced by f.

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where the maps $\beta_\bullet$ are given by a choice of basis and Φ is the matrix of f with respect to the chosen bases.

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where the maps $\beta_\bullet$ are given by a choice of basis and Φ is the matrix of f with respect to the chosen bases.

We can do pretty much the same thing for $R = \mathbb{Z}[G]$, although we lose the fact that $[M]_{\mathbb{Z}[G]}$ is a free rank one $\mathbb{Z}[G]$-module.

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Proposition (*Chinburg*)

Suppose $Cl(\mathcal{O}_S) = 1$. *There exists an exact sequence of* $\mathbb{Z}[G]$*-modules*

$$\tau_S : 0 \to U_S \to E_0 \xrightarrow{d} E_1 \to X_S \to 0$$

such that E_0, E_1 *are finitely generated of finite projective dimension.*

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Theorem

Suppose S contains the primes ramified in K/k. There exists

$$0 \to Cl(\mathcal{O}_S) \to \widetilde{X}_S \to X_S \to 0$$

such that we can take τ_S as above after replacing X_S by $\widetilde{X}_S$.

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τ_S gives rise to

$$\mathcal{E}_1 : 0 \to \mathbb{Q}U_S \to \mathbb{Q}E_0 \to \mathbb{Q}d(E_0) \to 0$$

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$$\iota : [\mathbb{Q}E_0]_{\mathbb{Q}[G]}[\mathbb{Q}E_1]^{-1}_{\mathbb{Q}[G]} \xrightarrow{\mathrm{ev}_{\mathbb{Q}d(E_0)}\circ(\iota(\mathcal{E}_1)\otimes\iota(\mathcal{E}_2))} [\mathbb{Q}U_S]_{\mathbb{Q}[G]}[\mathbb{Q}\widetilde{X}_S]^{-1}_{\mathbb{Q}[G]}.$$

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Lastly we take the $\mathbb{R}[G]$-module isomorphism ξ_S to be

$$\xi_S : [\mathbb{R}E_0]_{\mathbb{R}[G]}[\mathbb{R}E_1]^{-1}_{\mathbb{R}[G]} \xrightarrow{\mathbb{R}\otimes\iota} [\mathbb{R}U_S]_{\mathbb{R}[G]}[\mathbb{R}\widetilde{X}_S]^{-1}_{\mathbb{R}[G]} \xrightarrow{t(\lambda_S)} \mathbb{R}[G].$$

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Definition (*Determinant lattice*)

$$\Xi_S = \xi_S([E_0]_{\mathbb{Z}[G]}[E_1]^{-1}_{\mathbb{Z}[G]}).$$

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The determinant lattice Ξ_S will be a prediction of a lattice which encodes the leading term of $L_S(\chi, s)$ at $s = 0$. To define this lattice we need:

Definition

For irreducible representations χ of G, define $e_\chi \in \mathbb{C}[G]$ to be the central idempotent given by

$$e_\chi(\rho) = \begin{cases} \chi, & \text{if } \rho = \chi \\ 0, & \text{else.} \end{cases}$$

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Definition (*Stickelberger element*)

Define $\theta_S(s) = \sum_{\chi \in \hat{G}} L(\overline{\chi}, s) e_\chi$ and write $\theta_S^*(0)$ for the leading term at $s = 0$.

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Theorem

This conjecture is known to hold for

1. $k = \mathbb{Q}$ *(Burns, Greither, Flach);*
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This is supposed to be a 'universal' refinement of Stark's conjecture, which in turn was a 'weak' generalisation of the analytic class number formula.

Let's now try to understand how this relation works.

Stark's Conjectures and the eTNC

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Take $K = k$, so G is trivial. There is a unique class of 2-extensions and we have $\widetilde{X}_S \cong X_S \times \mathrm{Cl}(\mathcal{O}_S) \cong \mathbb{Z}^{|S|-1} \times \mathrm{Cl}(\mathcal{O}_S)$,

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$$\tau_S : 0 \to U_S \cong \mathbb{Z}^{|S|-1} \times \mu(k) \to E_0 \xrightarrow{0} E_1 \to \widetilde{X}_S \to 0,$$

with $E_0 = \mathbb{Z}^{|S|-1} \times \mu(k)$ and $E_1 = \mathbb{Z}^{|S|-1} \times \mathrm{Cl}(\mathcal{O}_S)$.

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Hence we have

$$\xi_S : \frac{h_S}{w}[\mathbb{Z}^{|S|-1}]_{\mathbb{Z}}[\mathbb{Z}^{|S|-1}]_{\mathbb{Z}}^{-1} \xrightarrow{\mathbb{R}\otimes\iota(\mathcal{E}_1)\iota(\mathcal{E}_2)} [\mathbb{R}U_S]_{\mathbb{R}}[\mathbb{R}\widetilde{X}_S]_{\mathbb{R}}^{-1} \xrightarrow{t(\lambda_S)} \mathbb{R}[G].$$

Stark's Conjectures and the eTNC

Therefore, the eTNC gives

$$\mathbb{Z} \cdot \theta_S^*(0) = \Xi_S = \frac{h_S \det(\lambda_S)}{w} \cdot \mathbb{Z},$$

and so the leading term of $\theta_S(0) = \zeta_S(0)$ is $\pm h_S \det(\lambda_S)/w$.

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This $\pm$ is the best we can hope for, because the eTNC is 'sensitive to changes in sign', while we took absolute values in the definition of Dirichlet's regulator.

Now let's re-cast Stark's conjecture in terms of $\theta_S^*(0)$.

Stark's Conjectures and the eTNC

Fix a $\mathbb{Q}[G]$-module isomorphism $f : \mathbb{Q}U_S \to \mathbb{Q}X_S$ and consider the quantity

$$R(f) = \det_{\mathbb{R}[G]}(\lambda_S \circ f^{-1}) \in \mathbb{R}[G]^\times.$$

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Proposition

Stark's main conjecture in the abelian setting is equivalent to the statement

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Stark's main conjecture in the abelian setting is equivalent to the statement

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The idea here is that we can identify $\mathbb{C}[G]$ with $\prod_\chi \mathbb{C}$. Then the statement becomes

$$\chi(\theta_S^*(0)R(f)^{-1})^\sigma = \chi^\sigma(\theta_S^*(0)R(f)^{-1}) \text{ for all } \chi,$$

for all $\sigma \in \mathrm{Aut}(\mathbb{C})$ – but we also find

$$\chi(\theta_S^*(0)/R(f)) = L(\chi)/\det(\lambda_S^{-1} \circ f \mid \chi) = A(\chi, f)^{-1}.$$

Stark's Conjectures and the eTNC

Upon tensoring with $\mathbb{Q}$, the eTNC gives

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Therefore we have

$$\text{eTNC} \implies \theta_S^*(0) \cdot R(f)^{-1} \in \mathbb{Q}[G] \implies \text{Stark's conjecture.}$$

What next?

So far we have

- Stated Stark's conjecture
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What does the eTNC actually do for us?

Well, let's rewind for a moment.

Why the plural?

Stark's conjectures are quite miraculous – particularly in the case where the order of vanishing of is 1, where there are many striking consequences of Stark's main conjecture.

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Suppose $S = T \cup \{\mathfrak{p}\}$ for $\mathfrak{p}$ totally split in K. It is a fact that $u\theta_T(0) \in \mathbb{Z}[G]$ for all $u \in \mathrm{Ann}_{\mathbb{Z}[G]}(\mu(K))$.

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Conjecture (*Brumer–Stark*)

Set

$$I_K^T := \{\mathfrak{I} \in I_K \mid \mathfrak{I}^{\theta_T(0)} = (u), \exists\, \varepsilon \colon Wu = \varepsilon \text{ in } \mathbb{Q}K^\times, K(\varepsilon^{1/W})/K \text{ is abelian}\}.$$

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We have

$$I_K^T = I_K.$$

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Note that this is a very boring statement if $\theta_S(0) = 0$. Let's try and generalise this for when $\theta_S(0)$ vanishes to higher powers.

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Macias Castillo showed that the answer is in the affirmative for K/k a quadratic extension, amongst some other progress.

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- The difficulty in verifying our case of the eTNC comes from computing the 2-extension τ_S.
- To state the full eTNC, replace 2-extensions τ_S by ‘perfect complexes’ over $\mathbb{Z}_p[G]$ for each prime p. This is hard!